About multigrid convergence of some length estimators*

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Abstract. An interesting property for curve length digital estimators is the convergence toward the continuous length and the associate convergence speed when the digitization step h tends to 0. On the one hand, it has been proved that the local estimators do not verify this convergence. On the other hand, DSS and MLP based estimators have been proved to converge but only under some convexity and smoothness or polygonal assumptions. In this frame, a new estimator class, the so called *semi-local estimators*, has been introduced by Daurat *et al.* in [4]. For this class, the pattern size depends on the resolution but not on the digitized function. The semi-local estimator convergence has been proved for functions of class $\mathcal{C}^2$ with an optimal convergence speed that is a $\mathcal{O}(h^{\frac{1}{2}})$ without convexity assumption. A semi-local estimator subclass, that we call *sparse estimators*, is exhibited here. The sparse estimators are proved to have the same convergence speed as the semi-local estimators under the weaker assumptions. Besides, if the continuous function that is digitized is concave, the sparse estimators are proved to have an optimal convergence speed in h. Furthermore, sparse estimation computational complexity in the optimal case is a $\mathcal{O}(h^{-\frac{1}{2}})$.

1 Introduction

The ability to perform the measurement of geometric features on digital representations of continuous objects is an important goal in a world becoming more and more digital. We focus in this paper on one classical digital problem: the length estimation. The problem is to estimate the length of a continuous curve S knowing a digitization of S. As information is lost during the digitization step, there is no reliable estimation without *a priori* knowledge and it is difficult to evaluate the estimator performances. In order to refine the evaluation of the estimators, a property, so called *convergence property* is desirable: the estimation convergence toward the true length of the curve S when the grid step h tends to 0. This property can be viewed as a robustness to digitization grid change. The local estimators based on a fixed pattern size do not satisfy the convergence property [13]. The adaptive estimators based on the Maximal Digital Straight

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Segment (MDSS) or the Minimum Length Polygon (MLP) satisfy the convergence property under assumptions of convexity, 4-connectivity for closed simple curves (also called Jordan curves) [**?**]. The semi-local estimators, introduced by Daurat et al [4] for function graphs, verifies the convergence property under smoothness assumption but without convexity hypothesis. We present here a subclass of the semi-local estimators, the *sparse estimators* that only need information on a small part of the function values and keep the convergence property. Moreover they have a convergence speed in h for smooth concave function.

The paper is organized as follows. In Section 2, some necessary notations and conventions are recalled, then existing estimators and their convergence properties are detailed. In Section 3, the sparse estimators are defined, their convergence properties are given in the general case and then in the concave cases (we make a distinction between the concavity of the continuous function and the concavity of the piecewise affine function related to the discretization). Section 4 concludes the article and gives directions for future works. Appendix A contains a few technical lemmas. In Appendix B two counterexamples about the concavity are exhibited. Appendix C presents a minimal error on the sparse estimation of the length of a segment of parabola.

2 Background

2.1 Discretization models

In this work, we have restricted ourselves to the digitizations of function graphs. So, let us consider a continuous function $g : [a, b] \to \mathbb{R}$ $(a < b)$, its graph $\mathcal{C}(g) = \{(x, g(x)) \mid x \in [a, b]\}$ and a positive real number r, the *resolution*. We assume to have an orthogonal grid in the Euclidean space $\mathbb{R}^2$ whose set of grid points is $h\mathbb{Z}^2$ where $h = 1/r$ is the *grid spacing*. We use the following notations: $[x]_h$ is the multiple of the grid spacing h that is the closest to x (with uppermost tie-breaking rule), $\lfloor x \rfloor_h$ is the greatest multiple of h less than or equal to x, $\{x\}_h = x - \lfloor x \rfloor_h$. For $i \leq j$ two integers, $[\![i, j]\!]$ stands for $[i, j] \cap \mathbb{Z}$. Finally, for any function f defined on an interval, $L(f)$ denotes the length of $\mathcal{C}(f)$, the graph of f $(L(f) \in [0, +\infty])$.

The common methods to model the digitization of the graph $\mathcal{C}(g)$ at the resolution r are closely related to each others.

In this paper, we assume an *object boundary quantization* (OBQ). This method associates to the graph $\mathcal{C}(g)$ the *h-digitization* set $\mathcal{D}^{\mathrm{O}}(g, h) = \{(kh, \lfloor g(kh) \rfloor_h) \mid k \in \mathbb{Z}\}$. The set $\mathcal{D}^{\mathrm{O}}(g, h)$ contains the uppermost grid points which lie in the hypograph of g, hence it can be understood as a part of the boundary of a solid object. Provided the slope of g is limited by 1 in modulus, $\mathcal{D}^{\mathrm{O}}(g, h)$ is an 8-connected digital curve. Observe that if g is a function of class C^1 such that the set $\{x \in [a, b] \mid |g'(x)| = 1\}$ is finite, then by symmetries on the graph $\mathcal{C}(g)$, it is possible to come down to the case where $|g'| \leq 1$. So, we assume that g is a Lipschitz function which Lipschitz constant 1. Hence, the set $\mathcal{D}^{\mathrm{O}}(g, h)$ is 8-connected for any h and the curve $\mathcal{C}(g)$ is rectifiable $(L(g) < +\infty)$. Moreover, the h-digitization set $\mathcal{D}^{\mathrm{O}}(g,h)$ can be described by its first point and its *Freeman code* [9], $\mathcal{F}(g,h)$, with the alphabet $\{0,1,7\}$. For any word $\omega \in \{0,1,7\}^k$ ($k \in \mathbb{N}$), we set $\|\omega\| = \sqrt{k^2+j^2}$ where j is the number of letters 1 minus the number of letters 7 in the word ω.

2.2 Local estimators

Local length estimators (see [10] for a short review) are based on parallel computations of the length of fixed size segments of a digital curve. For instance, an 8-connected curve can be split into 1-step segments. For each segment, the computation return 1 whenever the segment is parallel to the axes (Freeman's code is even) and $\sqrt{2}$ when the segment is diagonal (Freeman's code is odd). Then all the results are added to give the curve length estimation.

This kind of local computation is the oldest way to estimate the length of a curve and has been widely used in image analysis. Nevertheless, it has not the convergence property. In [13], the authors introduce a general definition of local length estimation with sliding segments and prove that such computations cannot give a convergent estimator for straight lines whose slope is small (less than the inverse of the size of the sliding segment). In [15], a similar definition of local length estimation is given with disjoint segments. Again, it is shown that the estimator failed to converge for straight lines (with irrational slopes). This behavior is experimentally confirmed in [3] on a test set of five closed curves. Moreover, the non-convergence is established in [5,16] for almost all parabolas.

2.3 Adaptative estimators: DSS and MLP

Adaptive length estimators gather estimators relying on a segmentation of the discrete curve that depends on each point of the curve: a move on a point can change the whole segmentation. Unlike to local estimators, it is possible to prove the convergence property of adaptive length estimators under some assumptions. Adaptive length estimators include two families of length estimators, namely the Maximal Digital Straight Segment (MDSS) based length estimators and the Minimal Length Polygon (MLP) based length estimators.

Definition and properties of MDSS can be found in [12,7,3]. Efficient algorithms have been developed for segmenting curves or function graphs into MDSS and to compute their characteristics in a linear time [12,8,7]. The decomposition in MDSS is not unique and depends on the start-point of the segmentation and on the curve travel direction. The convergence property has been proved for MDSS estimators with the following theorem that concerns closed curves:

Theorem 1 (MDSS estimators multigrid convergence [11, Theo. 13]). *Let S be a convex polygon whose h-digitization is connected for all $h \leq h_0$. Let $L(S)$ be the perimeter of S. Then, from a certain resolution, the MDSS length estimation L_{est} is such that*

$$|L(S) - L_{\mathrm{est}}(h)| \leq (2+\sqrt{2})\pi h.$$

Empirical MDSS multigrid convergence has also been tested in [3,6] on smooth nonconvex planar curves. The obtained convergence speed is a $\mathrm{O}(h)$ as in Theorem 1. Nevertheless it has not been proved under these assumptions. Furthermore, in the proof of [11, Theo. 13], we observe that the authors make explicitly the assumption that the discretization of the convex polygon is also a convex polygon[1]. We think that such an assumption is far from obvious since in the framework of local and semi-local estimation, we can prove that counterexamples exist (see Appendix B).

Let $\mathcal{C}$ be a simple closed curve lying in-between two polygonal curves γ_1 and γ_2. Then there is a unique polygon, the MLP, whose length is minimal between γ_1 and γ_2. The length of the MLP can be used to estimate the length of the curve $\mathcal{C}$. At least two MLP based length estimators have been described and proved to be multigrid convergent for convex, smooth or polygonal, simple closed curves, the SB-MLP proposed by Sloboda *et al.* [14] and the AS-MLP, introduced by Asano *et al.* [1]. For both of them, and for a given grid spacing h, the error between the estimated length L_{est} and the true length of the curve $L(\mathcal{C})$ is a $\mathcal{O}(h)$:

$$|L(C) - L_{\text{est}}(h)| \leq Ah$$

where $A = 8$ for SB-MLP and $A \approx 5.844$ for AS-MLP.

As estimators described in this section are adaptive, the asymptotic convergence is difficult to assess and the convergence theorems rely on strong hypotheses: convex polygonal curve or, for SB-MLP, convex polygonal or smooth curve with a 4-connected h-digitization below some grid spacing h_0. Moreover they suppose a sequential processing: each pattern (or vertex) depends on the already defined patterns (or vertices). The study of the MDSS in [6] shows that the MDSS size tends to 0 and their discrete length tends toward infinity. In order to keep the advantage of the adaptive estimators without their drawback, it is possible to define estimators with a non-adaptive pattern length that tends to 0 and such that its discrete length tends toward infinity when h tends to 0.

2.4 Semi-local length estimators

The notion of semi-local estimator appears in [4]. At a given resolution, a semi-local estimator resembles a local estimator: it can be implemented via a parallel computation, each processor handling a fixed size segment of the curve. Nevertheless, in the framework of semi-local estimation, the processors must be aware of the resolution from which the size of the segments depends.

[1] Though the digitization of a convex set is digitally convex, it does not mean that a polygonal curve related to a convex polygonal curve via a MDSS segmentation process is also convex.

More formally, let $g\colon [a,b] \to \mathbb{R}$ be a 1-Lipschitz function[2]. Hence, at any resolution, the Freeman's code describing the discretization of g belongs to the set $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \{0,1,7\}^n$.

A *semi-local estimator* is a pair (H,p) where

- $H\colon]0,\infty[\to \mathbb{N}^*$ gives the relative size of the segments given a grid spacing h and
- $p\colon \mathcal{P} \to [0,\infty[$ gives the estimated feature (here, the length) associated to a (finite) Freeman's code.

At a given grid spacing h, the Freeman's code describing the digitization of the curve $\mathcal{C}(g)$ is segmented in N_h codes ω_i of length $H(h)$ and a rest $\omega_* \in \{0,1,7\}^j$, $j < H(h)$. Then, the length of the curve $\mathcal{C}(g)$ is estimated by

$$L^{\mathrm{SL}}(g,h) = h \sum_{i=0}^{N_h-1} p(\omega_i).$$

In [4], the authors give a proof of convergence for functions of class C^2.

Theorem 2 ([4, Prop. 1]). *Let (H,p) be a semi-local estimator such that:*

1. $\lim_{h\to 0} hH(h) = 0$,
2. $\lim_{h\to 0} H(h) = +\infty$,
3. $\max\left\{p(\omega) - \|\omega\| \mid \omega \in \{0,1\}^k\right\} = \mathrm{o}(k)$ *as* $k \to +\infty$.

Then, for any function $g \in C^2([a,b])$, the estimation $L^{\mathrm{SL}}(g,h)$ converge toward the length of the curve $\mathcal{C}(g)$. Furthermore, if the term $\mathrm{o}(k)$ in the third hypothesis is a constant and $H(h) = \Theta(h^{-\frac{1}{2}})$, then $L(g) - L^{\mathrm{SL}}(g,h) = \mathcal{O}(h^{\frac{1}{2}})$.

$H(h)$ stands for the size of a Freeman's code ω while $hH(h)$ is the real length of the computation step. In the above theorem, the first hypothesis states that the real length $hH(h)$ tends to 0. If instead of diminishing the grid spacing, we keep it constant while doing a magnification of the curve with a factor $1/h$, the second hypothesis states that the size $H(h)$ of a code tends to infinity. Finally, and informally speaking, the last hypothesis states that the function p applied to a code ω must return a value close to the diameter[3] of the subset of $\mathcal{D}^{\mathrm{O}}(g,h)$ associated to ω.

3 Sparse estimators

In this section, we introduce a new notion, derived from semi-local estimators. Yet, on the contrary to semi-local estimators, we discard the information given by the codes ω_i but their extremities. It is as if we had two resolutions, one for the space (the abscissas), one for the calculus (the ordinates).

[2] In [4], the hypothesis on g is not clear. On the one hand, the code $\mathcal{F}(g,h)$ is supposed to have $\{0,1\}$ as alphabet. On the other hand, [4, Prop 1] does not retain any hypothesis on g but its class of differentiability. Indeed, in the proof, the derivative of g needs not be positive nor limited by 1.

[3] The maximal Euclidean distance between two points of the subset.

3.1 Definition

Definition 1. *Let* $H\colon]0,+\infty[\to \mathbb{N}^*$ *such that* $\lim_{h\to 0} H(h) = +\infty$ *and* $\lim_{h\to 0} hH(h) = 0$*. We say that* H *is* sparsity function*. Let* $g\colon [a,b] \to \mathbb{R}$ *be a rectifiable function. The* H-sparse estimator *of the length of the curve* $\mathcal{C}(g)$ *is defined by*

$$L^{\mathrm{Sp}}(g,h) = h \sum_{i=0}^{N_h} \|\omega_i\|$$

where $\omega_i \in \{0,1,7\}^{H(h)}$ *for* $i \neq N_h$*,* $\omega_{N_h} \in \{0,1,7\}^j$ *with* $j \in [0, N_h)$ *and the concatenation of the words* ω_i *equals* $\mathcal{F}(g,h)$*.*

An Illustration is given Figure 1.

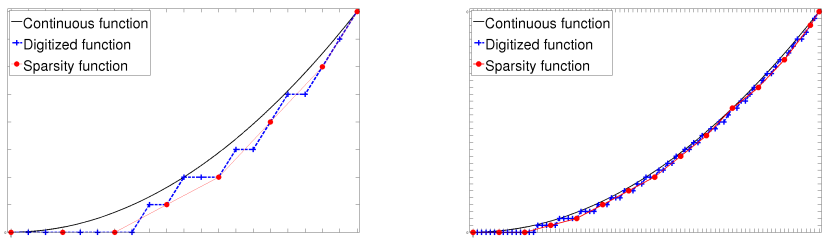


Fig. 1: Sparse estimation at two resolutions

3.2 Convergence

In this section, we establish that the sparse length estimators are convergent for Lipschitz functions. Moreover, Theorem 3 gives a bound on the error at grid spacing h for differentiable functions whose derivative is Lipschitz continuous.

Notations In the remainder of the report, we use the following notations. Let $h > 0$. We set $m = hH(h)$. The integers A, B are resp. the minimum and the maximum of the integer interval $\{k \in \mathbb{N} \mid kh \in [a,b]\}$. The functions g_a, g_h, g_b are resp. the restrictions of the function g to the intervals $[a, Ah]$, $[Ah, Bh]$, $[Bh, b]$ and $\sigma_h = (x_i)_{i=0}^N$ is the partition of $[Ah, Bh]$ defined by $x_i = Ah + im$ if $Ah + im < Bh$ and $x_N = Bh$. Note that A, B, N actually depend on h. We also define the function $\varphi\colon \mathbb{R} \to \mathbb{R}$ by $\varphi(x) = \sqrt{1+x^2}$. Thus, when g is of class C^1, one has $L(g) = \int_a^b \varphi \circ g'(t)\,\mathrm{d}t$.

The proof of Theorem 3 can be split in three parts. The first one gives a bound on the error due to the ignorance of the exact abscissas of the curve extremities. The second one evaluates the difference between the length of the curve $\mathcal{C}(g_h)$ and the length of the curve of the piecewise affine function g_m defined on $[Ah, Bh]$ by $g_m(x_k) = g(x_k)$ $(0 \leq k \leq N)$. The third part evaluates the difference between $\mathrm{L}(g_m)$ and the length of the piecewise affine function g_m^h defined on $[Ah, Bh]$ by $g_m^h(x_k) = \lfloor g_m(x_k) \rfloor_h = \lfloor g(x_k) \rfloor_h$ $(0 \leq k \leq N)$. Figure 2 shows the three functions g, g_m, g_m^h on an interval $[x_k, x_{k+1}]$.

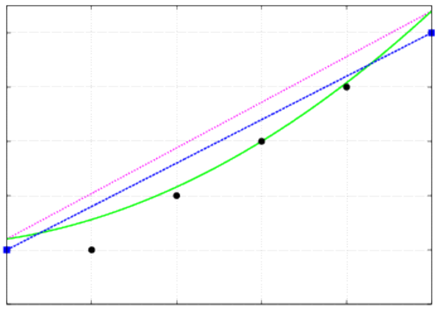


Fig. 2: The two main parts of the estimation error: the curve g (in green, solid) to its chord g_m (in magenta, dotted-dashed) then the curve chord to the chord g_m^h (in blue, dashed) of the digitized curve $\mathcal{D}^{\mathrm{O}}(g,h)$ (black points).

Sparse estimators can deal with discrete curves that are not 8-connected. That is why in the following propositions, we just assume that the function g is Lipschitz without any further hypothesis on the Lipschitz constant.

Lemma 1. *For any sparsity function H and any Lipschitz function g, we have*

$$\lim_{h \to 0} L(g_m) = L(g).$$

Furthermore, $L(g) \leq \varphi(k)(b-a)$ where k is a Lipschitz constant for g.

Proof. Since g is k-Lipschitz, the slope of any chord of $\mathcal{C}(g)$ is less than k in modulus. It follows that the length of any polyline fitting $\mathcal{C}(g)$ is bounded by $\varphi(k)(b-a)$. Then, according to Jordan's definition of arc length, we get

$$L(g) \leq \varphi(k)(b-a).$$

For any partition σ of an interval I, we note L_σ the length of the polyline associated to the partition. Remember that, from the Jordan's definition of the arc length and the triangle inequality, if σ and σ' are two partitions of the interval I such that $\sigma \subseteq \sigma'$ and f is a rectifiable function defined on I, then $L_\sigma \leq L_{\sigma'} \leq L(f)$.

Let $\varepsilon > 0$ and $\sigma_0 = (y_i)_{i=0}^n$ be a partition of $[a,b]$ such that

$$L(g) - L_{\sigma_0} < \varepsilon/4.$$

Since H is a sparsity function, there exists h_0 such that

$$0 < h_0 < \frac{\varepsilon}{8\varphi(k)}$$

and

$$\forall u \in (0, h_0),\ uH(u) < \min\Bigg(\frac{\varepsilon}{2(n-1)(\varphi(k)-1)}, \\ \min\{y_{i+1} - y_i \mid 0 \leq i \leq n-1\}\Bigg)$$

Let $h \in (0, h_0)$. We set

$$\sigma_{0,h} = (\sigma_0 \cup \sigma_h) \cap [Ah, Bh],$$
$$\sigma_a = (\sigma_0 \cap [a, Ah]) \cup \{Ah\}$$

and

$$\sigma_b = (\sigma_0 \cap [Bh, b]) \cup \{Bh\}.$$

Then we define $\sigma_1 = \sigma_a \cup \sigma_{0,h} \cup \sigma_b$. Firstly, we observe that

$$\begin{aligned}
L(g) - L_{\sigma_{0,h}} &\leq (L(g) - L_{\sigma_1}) + (L_{\sigma_1} - L_{\sigma_{0,h}}) \\
&\leq (L(g) - L_{\sigma_0}) + L_{\sigma_a} + L_{\sigma_b} \\
&\leq \frac{\varepsilon}{4} + 2\varphi(k)h \qquad \text{(for } g \text{ is } k\text{-Lipschitz)} \qquad (1)\\
&\leq \frac{\varepsilon}{2} \qquad\qquad \text{(for } h < h_0 < \frac{\varepsilon}{8\varphi(k)}\text{)}
\end{aligned}$$

Then we give an upper bound for $L_{\sigma_{0,h}} - L_{\sigma_h}$. For any i in $[\![1, N-1]\!]$ such that $y_i \in (Ah, Bh)$, there exists an integer that we note $s(i)$ s.t. $y_i \in [x_{s(i)}, x_{s(i+1)})$. Since $m < y_{i+1} - y_i$ for any $y_i, y_{i+1} \in (Ah, Bh)$, one has $s(i) < s(i+1)$. Let P_i, Q_i and R_i be respectively the points of $\mathcal{C}(g)$ with abscissas $x_{s(i)}$, y_i, $x_{s(i)+1}$. We have (remember that $x_{s(i)+1} - x_{s(i)} = m$),

$$\begin{aligned}
L_{\sigma_{0,h}} - L_{\sigma_h} &= \sum_{y_i \in (Ah,Bh)} \mathrm{d}(P_i, Q_i) + \mathrm{d}(Q_i, R_i) - \mathrm{d}(P_i, R_i) \\
&\leq \sum_{y_i \in (Ah,Bh)} (\varphi(k) - 1)m \qquad \text{(for } g \text{ is } k\text{-Lispschitz)} \\
&\leq (n-1) \times (\varphi(k) - 1)m \\
&\leq (n-1) \times (\varphi(k) - 1)\, \frac{\varepsilon}{2(n-1)(\varphi(k)-1)} \\
&\leq \frac{\varepsilon}{2}.
\end{aligned}$$

Then,

$$\begin{aligned}
L(g) - L_{\sigma_h} &\leq (L(g) - L_{\sigma_{0,h}}) + (L_{\sigma_{0,h}} - L_{\sigma_h}) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&\leq \varepsilon.
\end{aligned}$$

We conclude the proof straightforwardly. □

From Lemma 1 we derive immediately a bound on the errors due to the loss of the true left and right extremities of the curve $\mathcal{C}(g)$.

Corollary 1. *For any k-Lipschitz function g, we have*

$$L(g_a) + L(g_b) \leq 2\varphi(k)h.$$

When g is differentiable and its derivative is Lipschitz continuous, the next lemma gives us a bound on the difference between the length of the curve $\mathcal{C}(g_h)$ and the length of the polyline $\mathcal{C}(g_m)$.

Lemma 2. *If g is differentiable and its derivative is Lipschitz continuous, we have for any $h > 0$*

$$L(g_h) - L(g_m) \leq \frac{M(b-a)}{2} m \tag{2}$$

where M is a Lipschitz constant for g'.

Proof. Let M be a Lipschitz constant for g'. Since the function φ is 1-Lipschitz continuous, the function $\psi = \varphi \circ g'$ is M-Lipschitz continuous. Thanks to the mean value theorem, we can find a sequence $(t_k)_{k=0}^{N-1}$ such that

$$\text{for any } k \in [\![0, N-1]\!],\ g'(t_k) = \frac{g(x_{k+1}) - g(x_k)}{x_{k+1} - x_k}.$$

Then,

$$\begin{aligned}
L(g_h) - L(g_m) &= \int_{Ah}^{Bh} \psi(t)\,\mathrm{d}t - \sum_{k=0}^{N-1} \sqrt{(g(x_{k+1}) - g(x_k))^2 + (x_{k+1} - x_k)^2} \\
&= \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} \psi(t) - \psi(t_k)\ \mathrm{d}t \\
&\leq \sum_{k=0}^{N-1} M \frac{(x_{k+1} - x_k)^2}{2} \\
&\leq \frac{M}{2} \sum_{k=0}^{N-1} (x_{k+1} - x_k) m \\
&\leq \frac{M(b-a)}{2} m
\end{aligned}$$

□

A trivial maximization shows that the absolute difference between $L(g_m)$ and $L(g_m^h)$ is bounded by $2Nh$. The next lemma gives us a better bound and provides a relation that will serve as a starting point when we will consider the case of concave functions.

Lemma 3. *Let f_1 and f_2 be two piecewise affine functions defined on $[c,d] \subset \mathbb{R}$ ($d > c$) with a common partition having p steps. Suppose that $f_1 \leq f_2$ and $\|f_1 - f_2\|_\infty \leq e$ for some $e \in \mathbb{R}$. Then*

$$|L(f_1) - L(f_2)| \leq p\, e.$$

Proof. Let $\sigma = (x_i)_{i=0}^{p}$ be the common partition for f_1 and f_2. We write m_i for $x_{i+1} - x_i$ and $s_{1,i}$, resp. $s_{2,i}$, for the slope of f_1, resp. f_2, on the interval $[x_i, x_{i+1}]$.

$$\begin{aligned} L(f_1) - L(f_2) &= \sum_{i=0}^{p-1} m_i(\varphi(s_{1,i}) - \varphi(s_{2,i})) \\ &= \sum_{i=0}^{p-1} \frac{s_{1,i} + s_{2,i}}{\varphi(s_{1,i}) + \varphi(s_{2,i})}\, m_i(s_{1,i} - s_{2,i}). \end{aligned}$$

Note that, for any $i < p$,

$$m_i(s_{1,i} - s_{2,i}) = f_2(x_{i+1}) - f_1(x_{i+1}) - (f_2(x_i) - f_1(x_i)).$$

Thus, as, by hypothesis, $f_1 \leq f_2$ and $\|f_1 - f_2\|_\infty \leq e$, we get

$$-e \leq m_i(s_{1,i} - s_{2,i}) \leq e.$$

Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $\rho(x) = \frac{x}{\varphi(x)}$. We observe that, for any $i \in [\![0, p-1]\!]$,

$$\min(\rho(s_{1,i}), \rho(s_{2,i})) \leq \frac{s_{1,i} + s_{2,i}}{\varphi(s_{1,i}) + \varphi(s_{2,i})} \leq \max(\rho(s_{1,i}), \rho(s_{2,i})).$$

Thus, as ρ is continuous, for any $i \in [\![0, p-1]\!]$ there exists a real $s_{0,i}$ between $s_{1,i}$ and $s_{2,i}$ such that

$$\rho(s_{0,i}) = \frac{s_{1,i} + s_{2,i}}{\varphi(s_{1,i}) + \varphi(s_{2,i})}.$$

Thereby, we have

$$L(f_1) - L(f_2) = \sum_{i=0}^{p-1} \rho(s_{0,i})\, m_i(s_{1,i} - s_{2,i}). \tag{3}$$

As $\|\rho\|_\infty = 1$ we conclude that

$$|L(f_1) - L(f_2)| \leq \sum_{i=0}^{p-1} |m_i(s_{1,i} - s_{2,i})| \leq pe.$$

□

Thanks to Lemma 1, Lemma 2, and Lemma 3, which is applied to the piecewise affine functions g_m and g_m^h (taking $e = h$), we can state our first theorem on the convergence of Sparse length estimators.

Theorem 3. *Let H be a sparsity function and $g\colon [a,b] \to \mathbb{R}$ a Lipschitz function. Then, the estimator L^{Sp} converges toward the length of the curve $\mathcal{C}(g)$. Furthermore, if g is differentiable and its derivative is Lipschitz continuous , we have*

$$L(g) - L^{\mathrm{Sp}}(g,h) \leq 2\|\psi\|_\infty h + \frac{b-a}{2} M\, hH(h) + (b-a)\frac{1}{H(h)} \tag{4}$$

where $\psi = \varphi \circ g'$ and M is a Lipschitz constant for g'.

The first term in the right side of Formula 4 is due to the error on the bounds of the curve. Indeed, the two bounds of the curve domain cannot be on the grid at any resolution when h tends to 0. Hence, we have an unavoidable error in $\mathcal{O}(h)$. Note that this error does not exist when one computes the length of the boundary of a solid object (for, in this case, the curve is closed). Formula 4 shows two opposite trends for the determination of the sparsity step $H(h)$: the term in $\mathcal{O}(hH(h))$ – the discretization error – corresponds to the curve sampling error and tends to reduce the step $H(h)$ while the term in $\mathcal{O}\left(\frac{1}{H(h)}\right)$ – the quantization error – corresponds to the error due to the quantization of the sample points and tends to extend the step. The optimal convergence speed in $h^{\frac{1}{2}}$ is then obtained taking $H(h) = \Theta(h^{-\frac{1}{2}})$. Thus, only one in about $h^{-\frac{1}{2}}$ value is needed to make a sparse estimation (which justifies the adjective sparse). Then, the complexity in the optimal case is a $\mathcal{O}(r^{\frac{1}{2}})$.

3.3 Concave functions

In this section, we assume that the function g is concave on $[a,b]$. This implies in particular that g admits left and right derivatives, noted g'_l and g'_r, at any point of (a,b) and is Lipschitz continuous on any closed subinterval of (a,b). Under this new hypothesis, we can improve the bound on the convergence speed of the estimated length toward the true length of the curve $\mathcal{C}(g)$. The functions g_m and g_m^h are those defined in Section 3.2. Lemmas 4 and 5 are improvements of Lemmas 2 and 3 for concave curves. Figure 3 shows some experiments that illustrate the convergence rate obtained with Theorem 4.

Lemma 4. *If g admits a right derivative in Ah and a left derivative in Bh and if there exists a real $k > 0$ such that $g'_r(x) - g'_l(y) \leq k(y-x)$ for any x, y such that $Ah \leq x < y \leq Bh$ then*

$$L(g_h) - L(g_m) \leq \frac{k(b-a)}{4}\, m^2. \tag{5}$$

Proof. We define the piecewise affine function $g_m^+\colon [Ah, Bh] \to \mathbb{R}$ by

$$g_m^+(x) = \min\bigl(g(x_i) + g'_r(x_i)(x - x_i),\ g(x_{i+1}) + g'_l(x_{i+1})(x - x_{i+1})\bigr)$$

where $[x_i, x_{i+1}]$ is the subinterval of the partition σ_h that contains x. Note that $gm^+(x_i)$ (resp. $gm^+(x_{i+1})$) is uniquely defined and is equal to $g(x_i)$ (resp.

$g(x_{i+1})$). Since g is concave, $g_m \le g_h \le g_m^+$ so we can apply Lemma 7 on each subinterval of the partition σ_h. Together with the hypothesis on the derivatives of g, this leads to the following inequalities.

$$\begin{aligned}
L(g_h) - L(g_m) &\le \sum_{i=0}^{N-1} (x_{i+1} - x_i)\, \frac{(g_r'(x_i) - g_l'(x_{i+1}))^2}{4} \\
&\le \sum_{i=0}^{N-1} \frac{k}{4}(x_{i+1} - x_i)^3 \\
&\le \frac{k(b-a)}{4} m^2.
\end{aligned}$$

Hence, the result holds. □

Observe that Inequality (5) corresponds to Inequality (2) that have been improved with m becoming m^2 under concavity assumption.

Lemma 5. *Let f_1 and f_2 be two piecewise affine functions defined on $[c,d] \subset \mathbb{R}, (c < d)$, with a common partition σ having p steps and such that $f_1 \le f_2 \le f_1 + e$ for some constant $e > 0$. If furthermore f_2 is concave, then*

$$|L(f_1) - L(f_2)| \le \frac{p}{M_h^\sigma} e^2 + Ue.$$

where M_h^σ is the harmonic mean of the lengths of σ subintervals and $U =$ $\max(\varphi'(s_{2,0}), \varphi'(s_{2,0}) - 2\varphi'(s_{2,p-1})))$ is a constant which depends on the slopes $s_{2,0}$ and $s_{2,p-1}$ of the first and the last segments of f_2.

Proof. Let $\sigma = (x_i)_{i=0}^p$ be the common partition for f_1 and f_2. We write m_i for $x_{i+1} - x_i$ and $s_{1,i}$, resp. $s_{2,i}$, for the slope of f_1, resp. f_2, on the interval $[x_i, x_{i+1}]$. Also, we recall that in Lemma 3 we proved (with weaker hypotheses) that

$$\begin{aligned}
L(f_1) - L(f_2) &= \sum_{i=0}^{p-1} \rho(s_{0,i})\, m_i(s_{1,i} - s_{2,i}) \\
&= \sum_{i=0}^{p-1} \rho(s_{2,i})\, m_i(s_{1,i} - s_{2,i}) + \sum_{i=0}^{p-1} (\rho(s_{0,i}) - \rho(s_{2,i}))\, m_i(s_{1,i} - s_{2,i})
\end{aligned}$$

where $s_{0,i}$ is between $s_{1,i}$ and $s_{2,i}$ and $\rho(x) = \frac{x}{\varphi(x)} = \frac{x}{\sqrt{1+x^2}} = \varphi'(x)$.

On the one hand, since the function f_2 is concave, the sequence $(s_{2,i})_{i=0}^{p-1}$ is decreasing as is the sequence $(\rho(s_{2,i}))_{i=0}^{p-1}$ (for the function ρ is increasing). Hence, we can apply Lemma 8 with the settings

$$\begin{aligned}
c_i &= m_i(s_{1,i} - s_{2,i}) \\
&= (f_1(x_{i+1}) - f_2(x_{i+1})) - (f_1(x_i) - f_2(x_i)), \\
u_i &= \rho(s_{2,i}) - \rho(s_{2,p-1}), \\
I &= [-e, e].
\end{aligned}$$

Then, we get

$$\begin{aligned}
\left|\sum_{i=0}^{p-1} \rho(s_{2,i})\, m_i(s_{1,i}-s_{2,i})\right| &\le \left|\sum_{i=0}^{p-1} \rho(s_{2,i}) - \rho(s_{2,p-1})\, m_i(s_{1,i}-s_{2,i})\right| \\
&\qquad + \left|\sum_{i=0}^{p-1} \rho(s_{2,p-1})\, m_i(s_{1,i}-s_{2,i})\right| \\
&\le (\rho(s_{2,0}) - \rho(s_{2,p-1}))e + |\rho(s_{2,p-1})|\, e \\
&\le U e.
\end{aligned}$$

where $U = \max(\rho(s_{2,0}), \rho(s_{2,0}) - 2\rho(s_{2,p-1}))$.

On the other hand, the function ρ is 1-Lipschitz, so we have

$$|\rho(s_{0,i}) - \rho(s_{2,i})| \le |s_{0,i} - s_{2,i}| \le |s_{1,i} - s_{2,i}|$$

.
Then

$$\begin{aligned}
\sum_{i=0}^{p-1} (\rho(s_{0,i}) - \rho(s_{2,i}))\, m_i(s_{1,i}-s_{2,i}) &\le \sum_{i=0}^{p-1} m_i(s_{1,i}-s_{2,i})^2 \\
&\le \sum_{i=0}^{p-1} \frac{(m_i(s_{1,i}-s_{2,i}))^2}{m_i} \\
&\le \frac{pe^2}{M_h^\sigma}.
\end{aligned}$$

Eventually, we get

$$|L(f_1) - L(f_2)| \le U e + \frac{p}{M_h^\sigma} e^2. \tag{6}$$

□

From Lemma 5 and Lemma 4, we derive the following bound on the speed of convergence when the function g is concave.

Theorem 4. *Let H be a sparsity function and $g\colon [a,b] \to \mathbb{R}$ a concave function of class C^2. Then, we have*

$$L(g) - L^{\mathrm{Sp}}(g,h) = \mathcal{O}(h^2 H(h)^2) + \mathcal{O}\left(\frac{1}{H(h)^2}\right).$$

Proof. On the one hand, since g is of class C^2, it satisfies the hypothesis of Lemma 4. So we have

$$L(g_h) - L(g_m) \le \frac{k(b-a)}{4}\,(hH(h))^2$$

where $k = \|g''\|_\infty$.

On the other hand, Lemma 5 applied with $f_1 = g_m^h$, $f_2 = g_m$, $p = N$ and $e = h$ gives

$$\left|L(g_m) - L(g_m^h)\right| \leq \frac{N}{M_h^\sigma} h^2 + Uh$$

where M_h^σ is the harmonic mean of the lengths of σ_h subintervals and the constant U can be taken as $\max(\varphi'(a), \varphi'(a) - 2\varphi'(b)))$. From $N \leq \frac{b-a}{hH(h)} + 1$ and $\frac{N}{M_h^\sigma} = (N-1)\frac{1}{hH(h)} + \frac{1}{Bh - x_{N-1}}$ we get

$$\left|L(g_m) - L(g_m^h)\right| \leq \frac{b-a}{H(h)^2} + h^2 + Uh.$$

Thus,

$$\left|L(g) - L(g_m^h)\right| \leq Th + \frac{k(b-a)}{4}(hH(h))^2 + \frac{b-a}{H(h)^2} + h^2 + Uh$$

where $T = 2\|\varphi \circ g'\|_\infty$.

Observing that either $(hH(h))^2 \geq h$ or $\frac{1}{H(h)^2} \geq h$, the result holds. □

The result given by Theorem 4 is illustrated on Fig. 3 with the natural logarithmic. Compared to Theorem 3, concavity allows squarring each term of the right hand side of the inequality, which does not change the optimal size for $H(h)$ but improves the optimal convergence speed up to h.

3.4 Strong concavity

When the function g is concave, the piecewise affine function g_m is clearly also concave. Nevertheless, the second piecewise function g_m^h is not necessary concave. Indeed, we exhibit in Appendix B a function g that is concave and for which the function g_m^h is nonconcave for any h below some threshold. This section gives some sufficient conditions for g_m^h to be also concave and studies the consequences on the convergence speed of such an assumption.

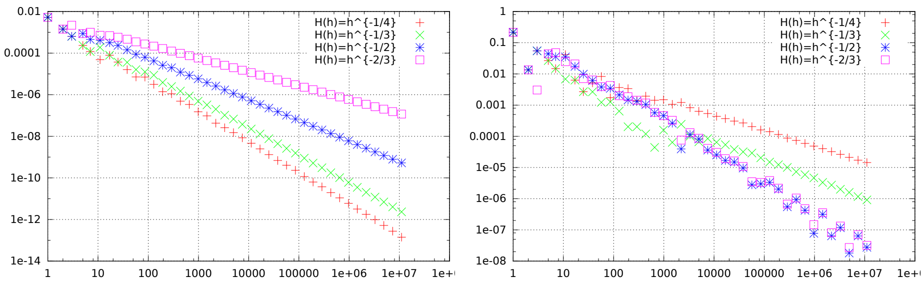
Proposition 1. *Let H be a sparsity function and $g\colon [a,b] \to \mathbb{R}$ a concave function of class C^2. If one of the following condition holds, then there exists $h_0 > 0$ such that, for any $h < h_0$, the piecewise affine function g_m^h is concave on $[Ah, (A + N_0 H(h))h]$ where $A = \left\lceil \frac{a}{h} \right\rceil$ and $N_0 = \left\lfloor \frac{b-a}{hH(h)} \right\rfloor$.*

(i) $H(h) = h^{-\frac{1}{2}}$ *and* $\max(g'') < -1$.
(ii) $H(h) = h^{-\frac{1}{2}}$ *and* $g(x) = ax^2 + bx + c$ *where* $|a| \geq \frac{1}{2}$.
(iii) $hH(h)^2 \to +\infty$ *as* $h \to 0$ *and* $\max(g'') < 0$.

Proof. The piecewise affine function g_m^h is concave on $[Ah, (A + N_0 H(h))h]$ iff $g_m^h(x_i + m) + g_m^h(x_i - m) - 2g_m^h(x_i) \leq 0$ for any $i \in [\![1, N_0 - 1]\!]$.

Let n be an integer in $[\![1, N_0 - 1]\!]$. We make Taylor expansions of g at x_n to the second order.

$$g(x_n + m) + g(x_n - m) = 2g(x_n) + m^2 g''(x_n) + o(m^2)$$



(a) $|L(g) - L(g_m)|$ (b) $\left|L(g_m) - L(g_m^h)\right|$

Fig. 3: An illustration of the various convergence rates. We have computed the length of the curve $y = \ln(x)$, $x \in [1, 2]$, using the sparse estimators defined by $H(h) = \lfloor h^{-\alpha} \rfloor$ where $\alpha \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$, for the resolutions defined by $r = \lfloor 1.5^n \rfloor$, $n \in [1, 40]$. (a) Discretization error (the errors on the left and the right bounds of the interval have been withdrew). We observe the convergence in $\mathcal{O}(h^2 H(h)^2)$ which appears in Theorem 4. (b) Quantization error. For $\alpha \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\}$, we observe the convergence is a $\mathcal{O}(1/H(h)^2)$, which appears in Theorem 4. For $\alpha = \frac{2}{3}$, the condition *(iii)* of Prop. 1 is satisfied and thus the piecewise affine function g_m^h is concave. Hence, we can observe that the convergence is a $\mathcal{O}(h)$ as deduced from Lemma 6.

Then,

$$\begin{aligned} g_m^h(x_n + m) + g_m^h(x_n - m) - 2g_m^h(x_n) & \\ &= g(x_n + m) + g(x_n - m) - 2g(x_n) + Eh \text{ where } |E| < 2 \\ &= m^2 g''(x_n) + Eh + o(m^2). \end{aligned}$$

Thus, setting $M = \max(g'')$,

$$\frac{g_m^h(x_n + m) + g_m^h(x_n - m) - 2g_m^h(x_n)}{h} < -hH(h)^2 M + 2 + o(hH(h)^2). \quad (7)$$

Thanks to Inequality (7), noting that its left hand side is an integer (also that when g is a 2-th order polynomial the term $o(hH(h)^2)$ vanishes), the reader can easily check the three parts of the proposition. $\square$

The following lemma is an improvement of Lemma 5 for two concave piecewise affine functions.

Lemma 6. *Let f_1 and f_2 be two concave piecewise affine functions defined on $[c, d] \subset \mathbb{R}$ such that $f_1 \leq f_2 \leq f_1 + e$ for some $e > 0$ and f_1, f_2 have the same monotonicity on each subinterval on which they are affine. Then*

$$|L(f_1) - L(f_2)| \leq Ue.$$

where U is a constant defined as follows. Given a common partition $\sigma = (x_i)_{i=0}^p$ of the interval $[c,d]$ related to the piecewise affine functions f_1 and f_2, let $s_{1,0}$, $s_{1,p-1}$, resp. $s_{2,0}$, $s_{2,p-1}$, be the slopes of the first and last segments of $\mathcal{C}(f_1)$, resp. $\mathcal{C}(f_2)$. Now, let $\alpha = \frac{s_{1,0}+s_{2,0}}{\varphi(s_{1,0})+\varphi(s_{2,0})}$ and $\beta = \frac{s_{1,p-1}+s_{2,p-1}}{\varphi(s_{1,p-1})+\varphi(s_{2,p-1})}$. Then,

$$U = \max(\alpha, \alpha - 2\beta).$$

Proof. Let $\sigma = (x_i)_{i=0}^p$ be a common partition for f_1 and f_2. We write m_i for $x_{i+1} - x_i$ and $s_{1,i}$, resp. $s_{2,i}$, for the slope of f_1, resp. f_2, on the interval $[x_i, x_{i+1}]$. Since f_1 and f_2 are concave, the sequences $(s_{1,i})$ and $(s_{2,j})$ are monotonically non-increasing. Furthermore, since f_1 and f_2 have the same monotonicity, $s_{1,i}$ and $s_{2,i}$ have the same sign for any $i \in [\![0, p-1]\!]$. In Lemma 3 we proved (with weaker hypotheses) that

$$L(f_1) - L(f_2) = \sum_{i=0}^{p-1} \rho(s_{0,i})\, m_i (s_{1,i} - s_{2,i}) \tag{8}$$

where $\rho(x) = \frac{x}{\varphi(x)} = \frac{x}{\sqrt{1+x^2}}$ and $\rho(s_{0,i}) = \frac{s_{1,i}+s_{2,i}}{\varphi(s_{1,i})+\varphi(s_{2,i})}$.

Firstly, we prove that the sequence $(\rho(s_{0,i}))$ is also monotonically non-increasing. Let $i < j$ be two integers in $[\![0, p-1]\!]$.

$$\begin{aligned}
\rho(s_{0,i}) \geq \rho(s_{0,j}) \iff& \frac{s_{1,i} + s_{2,i}}{\varphi(s_{1,i}) + \varphi(s_{2,i})} \geq \frac{s_{1,j} + s_{2,j}}{\varphi(s_{1,j}) + \varphi(s_{2,j})} \\
\iff& \big(s_{1,i}\varphi(s_{1,j}) - s_{1,j}\varphi(s_{1,i})\big) + \big(s_{2,i}\varphi(s_{2,j}) - s_{2,j}\varphi(s_{2,i})\big) \\
&+ \big(s_{1,i}\varphi(s_{2,j}) + s_{2,i}\varphi(s_{1,j})\big) \geq \big(s_{1,j}\varphi(s_{2,i}) + s_{2,j}\varphi(s_{1,i})\big) \\
\iff& \varphi(s_{1,i})\varphi(s_{1,j})\big(\rho(s_{1,i}) - \rho(s_{1,j})\big) \\
&+ \varphi(s_{2,i})\varphi(s_{2,j})\big(\rho(s_{2,i}) - \rho(s_{2,j})\big) \\
&+ \big(s_{1,i}\varphi(s_{2,j}) + s_{2,i}\varphi(s_{1,j})\big) \geq \big(s_{1,j}\varphi(s_{2,i}) + s_{2,j}\varphi(s_{1,i})\big)
\end{aligned}$$

The terms $\rho(s_{1,i}) - \rho(s_{1,j})$ and $\rho(s_{2,i}) - \rho(s_{2,j})$ are non-negative because $(s_{1,i})$ and $(s_{2,j})$ are monotonically non-increasing (and the function ρ is increasing). Hence,

$$\rho(s_{0,i}) \geq \rho(s_{0,j}) \Leftarrow s_{1,i}\varphi(s_{2,j}) + s_{2,i}\varphi(s_{1,j}) \geq s_{1,j}\varphi(s_{2,i}) + s_{2,j}\varphi(s_{1,i}) \tag{9}$$

If $s_{1,j} \geq 0$, and thus $s_{2,j}, s_{1,i}, s_{2,i} \geq 0$, we can square the two terms of the inequality in the right hand side of (9)

$$\begin{aligned}
\rho(s_{0,i}) \geq \rho(s_{0,j}) \Leftarrow& s_{1,i}^2(1 + s_{2,j}^2) + s_{2,i}^2(1 + s_{1,j}^2) + 2s_{1,i}s_{2,i}\varphi(s_{1,j})\varphi(s_{2,j}) \geq \\
& s_{1,j}^2(1 + s_{2,i}^2) + s_{2,j}^2(1 + s_{1,i}^2) + 2s_{1,j}s_{2,j}\varphi(s_{1,i})\varphi(s_{2,i}) \\
\Leftarrow& s_{1,i}^2 + s_{2,i}^2 + 2s_{1,i}s_{2,i}\varphi(s_{1,j})\varphi(s_{2,j}) \geq \\
& s_{1,j}^2 + s_{2,j}^2 + 2s_{1,j}s_{2,j}\varphi(s_{1,i})\varphi(s_{2,i}) \\
\Leftarrow& s_{1,i}^2 + s_{2,i}^2 + A\,\rho(s_{1,i})\rho(s_{2,i}) \geq s_{1,j}^2 + s_{2,j}^2 + A\,\rho(s_{1,j})\rho(s_{2,j})
\end{aligned}$$

where $A = 2\varphi(s_{1,i})\varphi(s_{2,i})\varphi(s_{1,j})\varphi(s_{2,j})$ is clearly non-negative.
Since the function ρ is monotonically non-decreasing and odd, it is plain that the last inequality is true under all our assumptions.

The case where $s_{1,i} \leq 0$, and thus $s_{2,i}, s_{1,j}, s_{2,j} \leq 0$ is similar. The last case, where $s_{1,i} \geq 0 \geq s_{1,j}$, and thus $s_{2,i} \geq 0 \geq s_{2,j}$, is obvious. Thereby, we have proved that the sequence $(\rho(s_{0,i}))$ is monotonically non-decreasing.

Now, from Lemma 8, taking

$$\begin{aligned} c_i &= m_i(s_{1,i} - s_{2,i}) \\ &= (f_1(x_{i+1}) - f_2(x_{i+1})) - (f_1(x_i) - f_2(x_i)), \\ u_i &= \rho(s_{0,i}) - \rho(s_{0,p-1}) \text{ and} \\ I &= [-e, e], \end{aligned}$$

we derive from (8) that

$$\begin{aligned} |L(f_1) - L(f_2)| &\leq (\rho(s_{0,0}) - \rho(s_{0,p-1}))e + |\rho(s_{0,p-1})| \left| \sum_{i=0}^{p-1} m_i(s_{1,i} - s_{2,i}) \right| \\ &\leq (\rho(s_{0,0}) - \rho(s_{0,p-1}))e + |\rho(s_{0,p-1})|\, e \\ &\leq Ue \end{aligned}$$

where $U = \max(\rho(s_{0,0}), \rho(s_{0,0}) - 2\rho(s_{0,p-1}))$. □

Corollary 2. *Let H be a sparsity function and $g\colon [a,b] \to \mathbb{R}$ a concave function of class C^2. If, for some $h_0 > 0$, the function g_m^h is concave on $[Ah, (A + N_0 H(h))h]$ where $A = \left\lceil \frac{a}{h} \right\rceil$ and $N_0 = \left\lfloor \frac{b-a}{hH(h)} \right\rfloor$ for any $h < h_0$, then we have*

$$L(g) - L^{\mathrm{Sp}}(g, h) = \mathcal{O}(h^2 H(h)^2) + \mathcal{O}(h).$$

Proof. From Corollary 1, we have

$$L(g_a) + L(g_b) \leq 2\|\varphi \circ g'\|_\infty h. \tag{10}$$

From Lemma 4, we get

$$L(g_h) - L(g_m) \leq \frac{(b-a)\|g''\|_\infty}{4}(hH(h))^2. \tag{11}$$

Let $N_0 = \left\lfloor \frac{b-a}{hH(h)} \right\rfloor$. We write $g_{m|1}$ and $g_{m|2}$, resp. $g_{m|1}^h$ and $g_{m|2}^h$ for the restrictions of the function g_m, resp. g_m^h, to the intervals $[A, (A + N_0 H(h))h]$ and $[(A + N_0 H(h))h, Bh]$. The functions $g_{m|1}$ and $g_{m|1}^h$ are piecewise affine with subintervals of width $m = hH(h)$ while, if $N_0 \neq N$, the functions $g_{m|2}$ and $g_{m|2}^h$ are affine on an interval of width αh where α is an integer in $[1, m)$. It follows from Lemma 6 that

$$\left| L(g_{m|1}) - L(g_{m|1}^h) \right| \leq Uh \tag{12}$$

where U is bounded by $\max(\varphi'(g'(a)+1), \varphi'(g'(a)+1) - 2\varphi'(g'(b)-1))$ which does not depend on h. Indeed, with the notations of Lemma 6, $U = \max(\alpha, \alpha - 2\beta)$ where $\alpha = \frac{s_{1,0}+s_{2,0}}{\varphi(s_{1,0})+\varphi(s_{2,0})}$ lies between $\frac{s_{1,0}}{\varphi(s_{1,0})}$ and $\frac{s_{2,0}}{\varphi(s_{2,0})}$, that is between $\varphi'(s_{1,0})$ and $\varphi'(s_{2,0})$. On the one hand $\varphi'(s_{1,0})$ is lower than $\varphi'(g'(a))$ for φ' is monotonically increasing and g_m is concave. On the other hand, it can easily be proved that $s_{2,0} \leq s_{1,0} + \frac{1}{H(h)} \leq g'(a) + 1$. Hence,

$$\alpha \leq \varphi'(g'(a)+1).$$

Alike, we have

$$\beta \geq \varphi'(g'(b)-1)$$

and thus

$$U \leq \max(\varphi'(g'(a)+1), \varphi'(g'(a)+1) - 2\varphi'(g'(b)-1)).$$

Finally, we derive immediately from Lemma 3 that

$$\left| L(g_{m|2}) - L(g^h_{m|2}) \right| \leq h. \tag{13}$$

As

$$\begin{aligned} L(g) &= L(g_a) + L(g_h) + L(g_b), \\ L(g_m) &= L(g_{m|1}) + L(g_{m|2}) \text{ and} \\ L^{\mathrm{Sp}}(g,h) &= L(g^h_m) \\ &= L(g^h_{m|1}) + L(g^h_{m|2}), \end{aligned}$$

the result follows readily from eqs. (10) to (13). □

From Corollary 2, it follows that, to speed up the convergence, we shall take the smallest sparsity step $H(h)$ provided the hypothesis about the concavity is satisfied. According to Proposition 1, this should lead us to choose the function H such that H dominates $h^{-\frac{1}{2}}$ as $h \to 0$. For instance, we can take $H(h) = h^{-\frac{1}{2}-\varepsilon}$ where $\varepsilon > 0$ and $\varepsilon \approx 0$. Then, the convergence speed is $h^{1-2\varepsilon}$. Note that h is a minimal error bound that cannot be improved in the general case since for the function g defined by $g(x) = (\frac{19}{48})^2 - x^2$, $x \in [\frac{1}{16}, \frac{19}{48}]$, we have shown that $L(g) - L^{\mathrm{Sp}}(g,h) \geq 0.06h$ (see Appendix C).

4 Conclusion

In this article, we have studied some convergence properties of a class of semi-local length estimators in the concave and the general cases. These estimators need few information about the curve: the proportion of points of the curve used for the computation tends to 0 as the resolution tends toward infinity. That is why we propose to call them sparse estimators. In a future work, we plan to extend our estimators to the nD Euclidean space to compute k-volumes, $k < n$. We have also to study how the material presented in this article behave with Jordan curves obtained as boundary of solid objects through various discretization schemes. Furthermore, the definition of the sparse estimators relies on Jordan's definition for curve length. It would be interesting to keep the main idea from these estimators while relying on the more general definition of Minkowski (as in [2]). This could be more realistic in the framework of multigrid convergence, since physic objects cannot be considered as smooth (nor convex, etc.) at any resolution. Another extension of this work is to check whether the proofs of convergence obtained for sparse estimators can help to obtain new proofs for the convergence of adaptative length estimators as the MDSS. This could lead to the definition of a larger class of geometric feature estimators including sparse estimators and MDSS. Eventually, there is a need to find how to estimate the resolution of a given curve.

A Technical lemmas

Lemma 7. *Let ABC be a triangle in $\mathbb{R}^2$ ($A \neq C$) and $\Gamma \subset \mathbb{R}^2$ be a rectifiable curve from A to C included in the triangle ABC such that the set between the segment AC and the curve Γ is convex. Let $\mathfrak{B}$ be an orthonormal basis of $\mathbb{R}^2$ such that, in the coordinate system $(A, \mathfrak{B})$, the abscissa of C, noted m, is positive and the abscissa of B strictly lies between 0 and m. Let α, β, γ be the slopes, in the basis $\mathfrak{B}$, of the line from B to C, resp. from C to A, resp. from A to B. Then, the length of Γ, L_Γ, is such that*

$$AC \leq L_\Gamma \leq AC + m\frac{(\gamma - \alpha)^2}{4}.$$

Fig. 4 illustrates the configuration studied in Lemma 7.

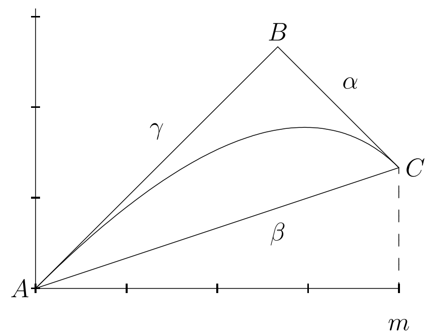


Fig. 4: α, β, γ are the slopes of the segments BC, CA, AB.

Proof. Since the set bounded by the segment AC and the curve Γ is convex and included in the triangle ABC, its perimeter is less than, or equal to, the perimeter of the triangle ABC (see [17, Part XII]). Thus, $AC \leq L_\Gamma \leq AB+BC$.

Since x_B, the abscissa of the point B in the coordinate system $(A, \mathfrak{B})$ verifies $0 < x_B < m$, β lies between α and γ. Then there exists a real $k \in [0,1]$ such that $\beta = k\gamma + (1-k)\alpha$. It can be seen that the vectors **AB**, **BC** and **AC** have coordinates $(km, km\gamma)$, $((1-k)m, (1-k)m\alpha)$ and $(m, m\beta)$. Thus,

$$\begin{aligned}
AB + BC - AC &= m\big(k\varphi(\gamma) + (1-k)\varphi(\alpha) - \varphi(\beta)\big) \\
&= m\Big(k\big(\varphi(\gamma) - \varphi(k\gamma + (1-k)\alpha)\big) + \\
&\qquad\qquad (1-k)\big(\varphi(\alpha) - \varphi(k\gamma + (1-k)\alpha)\big)\Big) \\
&= mk(1-k)(\gamma - \alpha)\big(\varphi'(\xi_1) - \varphi'(\xi_2)\big) \\
&= mk(1-k)(\gamma - \alpha)(\xi_1 - \xi_2)\varphi''(\xi).
\end{aligned}$$

where ξ_1, ξ_2, ξ lie between α and γ.

Hence,

$$AB + BC - AC \leq \frac{m(\gamma - \alpha)^2}{4} \tag{14}$$

for $\|\varphi''\|_\infty = 1$. So, the result holds.

$\square$

Remark 1. We could improve the previous result by a factor 2 since it appears from the above calculus that $AB + BC - AC$ is the 'vertical distance' between the function φ and one of its chord (see Fig. 5) and is thus maximal when the chord is 'horizontal'.

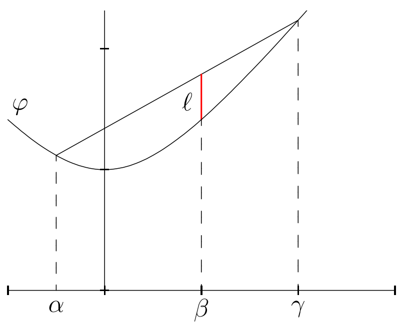


Fig. 5: $\ell = AB + BC - AC$.

Lemma 8. *Let $(u_n)_{n\in\mathbb{N}}$ a monotonically non-increasing sequence of real non negative numbers and $(c_n)_{n\in\mathbb{N}}$ a sequence of reals in an interval I such that $\sum_{i=0}^{j} c_i \in I$ for any integer j. Then, $\sum_{i=0}^{j} c_i\, u_i \in u_0\, I$ for any integer j.*

Proof. If $u_0 = 0$, then $u_n = 0$ for any n and the result is obvious. From now, we assume $u_0 > 0$. Let $n \in \mathbb{N}$ and $S = \sum_{i=0}^{n} c_i\, u_i$. We set $C_j = \sum_{i=0}^{j} c_i$ for any $j \le n$, $p_i = \frac{u_i - u_{i+1}}{u_0}$ for any $i \le n-1$ and $p_n = \frac{u_n}{u_0}$. The reals p_i are all non-negative and their sum equals 1. We can easily check that

$$\begin{aligned} S &= \sum_{i=0}^{n-1} \left(\sum_{j=0}^{i} c_j \right) (u_i - u_{i+1}) + \left(\sum_{j=0}^{n} c_j \right) u_n \\ &= u_0 \left(\sum_{i=0}^{n} p_i\, C_i \right) \end{aligned}$$

The last equality above shows that the real $\frac{1}{u_0} S$ is the barycenter –with non-negative weights– of numbers in the interval I. Thus, the result holds.

B Strong concavity: counterexamples

In this appendix, we show that a piecewise affine function can be concave and its digitization, beyond some resolution, never concave (that is, the piecewise affine function g_m^h defined in Sec. 3.2 is not concave for grid spacing h below some threshold). The first counterexample uses a local estimator and the second one uses a sparse estimator. Both counterexamples rely on the following theorem proved in [16] (in fact, an extended version of the theorem is needed for the second counterexample). This theorem asserts that, given a function $x \mapsto ax^2 + bx + c$, the distribution in $[0, h]$ of the values of the expression $\{a(kh)^2 + b(kh) + c\}_h$, $k \in \mathbb{N}$, which are the errors resulting from the quantization, tends toward the equidistribution.

Theorem 5 ([16, Lemma 2 and Prop. 3]). *Let $a, b \in \mathbb{R}$, $a < b$. Let $g : [a, b] \to \mathbb{R}$ be a polynomial function of degree 2. Then, for all interval $I \subseteq [0, 1]$,*

$$\lim_{h \to 0} \frac{\mathrm{card}\{x \in h\mathbb{N} \cap [a, b] \mid g(x) \bmod h \in hI\}}{\mathrm{card}(h\mathbb{N} \cap [a, b])} = \mu(I)$$

where $\mu(I)$ is the classical length of I.

B.1 Counterexample #1: local estimation

We digitize the parabola associated to the function $g(x) = 2x - x^2$, $x \in [0, 1]$ and we split this parabola into segments of size $5h$. Thanks to Theorem 5, we prove that, for each grid spacing h below some threshold, we can choose an integer p such that the finite difference $g_m^h((p+5)h) - g_m^h(ph)$ is less than or equal to the grid spacing h while the finite difference $g_m^h((p+10)h) - g_m^h((p+5)h)$ is greater than or equal to twice the grid spacing h. Thus, the function g_m^h is not concave on $[0, 1]$.

Detailed calculus
According to Theorem 5, it exists a real $h_0 > 0$ such that, for any $h \in (0, h_0)$, one has

$$\operatorname{card}\left\{n \in [\![\tfrac{103}{120h}, \tfrac{104}{120h}]\!] \mid g(nh) - g_m^h(nh) \in [\tfrac{4h}{12}, \tfrac{7h}{12})\right\} \geq \frac{1}{5}\operatorname{card}[\![\tfrac{103}{120h}, \tfrac{104}{120h}]\!].$$

It follows that there exists $h_1 > 0$ such that for any $h < h_1$, one can find $n_0 \in [\![\frac{103}{120h}, \frac{104}{120h}]\!]$ such that $n_0 + 10$ still lies in $[\![\frac{103}{120h}, \frac{104}{120h}]\!]$ and $g(n_0 h) - g_m^h(n_0 h) \in [\frac{4h}{12}, \frac{7h}{12})$. Then, noting that $\frac{16}{60} \leq g'(x) \leq \frac{17}{60}$ on $[\frac{103}{120}, \frac{104}{120}]$, we obtain

$$\begin{aligned} g_m^h((n_0+5)h) - g_m^h(n_0 h) &< g((n_0+5)h) - (g(n_0 h) - \frac{7}{12}h) \\ &< \frac{17}{60} \times 5h + \frac{7}{12}h \\ &< 2h. \end{aligned}$$

As the term in the left hand side of the above inequalities is a multiple of h, we get

$$g_m^h((n_0+5)h) - g_m^h(n_0 h) \leq h.$$

In the same way, we obtain

$$\begin{aligned} g_m^h((n_0+10)h) - g_m^h(n_0 h) &> g((n_0+10)h) - h - (g(n_0 h) - \frac{4}{12}) \\ &> \frac{16}{60} \times 10h - \frac{2}{3}h \\ &> 2h. \end{aligned}$$

Thus,

$$g_m^h((n_0+10)h) - g_m^h(n_0 h) \geq 3h.$$

Finally, we have

$$g_m^h((n_0+10)h) - g_m^h(n_0 h) > 2\big(g_m^h((n_0+5)h) - g_m^h(n_0 h)\big).$$

That is, the function g_m^h is strictly convex on $[n_0 h, (n_0+10)h]$.

B.2 Counterexample #2 : sparse estimation

For the second counterexample, we discretize the parabola $y = g(x) = \frac{1}{50}(2x - x^2)$, $x \in [0,1]$ and we use segments of size $H(h) = \left\lfloor h^{-\frac{1}{2}} \right\rfloor$. Substantially, this second counterexample is similar to the previous one though it requires the following extended version of Theorem 5 whose proof will be given in a future work.

Proposition 2. *Let $a, b \in \mathbb{R}$, $a < b$, and $g\colon [a,b] \to \mathbb{R}$ a quadratic polynomial function. Let $(J_h)_{h>0}$ a family of integer intervals such that $hJ_h \subseteq [a,b]$ for any $h > 0$ and $\lim_{h\to 0} \operatorname{card} J_h = +\infty$. Then, for any interval $I \subseteq [0,1]$, one has:*

$$\lim_{h \to 0} \frac{\operatorname{card}\{n \in J_h \mid g(nh) \bmod h \in hI\}}{\operatorname{card} J_h} = \mu(I)$$

C Inferior bound for the method error in the concave case

We give an inferior bound on the difference between the true length $L(g)$ of the parabola $y = g(x) = (\frac{19}{48})^2 - x^2$ for $x \in [\frac{1}{16}, \frac{19}{48}]$ and the length $L^{\mathrm{Sp}}(g,h)$, obtained with the sparse estimator defined by the sparsity function $H(h) = \left\lfloor h^{-\frac{1}{2}} \right\rfloor$. Let g_m and g_m^h be the piecewise affine functions defined in Section 3.2. Then the lengths of their curves satisfy $L(g_m^h) + 0.05h \leq L(g_m) \leq L(g)$ for any $h = (12(8p+1))^{-2}$ where $p \in \mathbb{N}$. Moreover, the bounds of the interval $[\frac{1}{16}, \frac{19}{48}]$ are multiple of h. Hence, there is no error due to the bounds. Eventually, for any $p \in \mathbb{N}$ and $h = (12(8p+1))^{-2}$, we get $L(g) - L^{\mathrm{Sp}}(g,h) \geq 0.05h$.

Detailed calculus
Let $h = \frac{1}{144(8p+1)^2}$ ($p \in \mathbb{N}$) and $H(h) = \left\lfloor h^{-\frac{1}{2}} \right\rfloor = 12(8p+1)$.
Thereby, here we have

- $m = hH(h) = \frac{1}{12(8p+1)}$,
- $Ah = \frac{1}{16}$ ($A = 9(8p+1)^2$), $Bh = \frac{19}{48}$ ($B = 57(8p+1)^2$),
- $N = \left\lceil \frac{\frac{19}{48} - \frac{1}{16}}{m} \right\rceil = \left\lceil \frac{1}{3} H_r \right\rceil = \frac{1}{3} H_r = 4(8p+1)$,
- For any $i \in [\![0, N]\!]$, $x_i = \frac{1}{16} + i\,m = x_0 + i\,m$ (in particular, the last interval of the sparse estimation have size m).

Furthermore, we have

$$\begin{aligned} g(x_0) &\equiv 0 \pmod{h} \\ \text{and} \quad g(x_i) &\equiv \frac{1}{2} i h \pmod{h} \end{aligned} \tag{15}$$

We set $c = \frac{h}{2} = \frac{m^2}{2}$, $z_i = \frac{1}{2}(x_i + x_{i+1})$ and $y_i = g(x_{i+1}) - g(x_i) = -2m\, z_i$.
Then, from (15), we derive

$$\begin{aligned} L(g_m) - L(g_m^h) = \sum_{i=0}^{16p+1} & \left(\sqrt{m^2 + {y_{2i}}^2} + \sqrt{m^2 + {y_{2i+1}}^2} \right) \\ & - \left(\sqrt{m^2 + (y_{2i} - c)^2} + \sqrt{m^2 + (y_{2i+1} + c)^2} \right) \end{aligned}$$

On the one hand

$$\begin{aligned} \sqrt{m^2 + {y_{2i}}^2} - \sqrt{m^2 + (y_{2i} - c)^2} &= -\frac{m^2}{4} \frac{8z_{2i} + m}{\sqrt{1 + 4{z_{2i}}^2} + \sqrt{1 + 4(z_{2i} + \frac{m}{4})^2}} \\ &\geq -\frac{m^2}{8} \frac{8z_{2i} + m}{\sqrt{1 + 4{z_{2i}}^2}} \end{aligned}$$

On the other hand

$$\sqrt{m^2 + {y_{2i+1}}^2} - \sqrt{m^2 + (y_{2i+1} + c)^2} = \frac{m^2}{4} \frac{8z_{2i+1} - m}{\sqrt{1 + 4{z_{2i+1}}^2} + \sqrt{1 + 4(z_{2i+1} - \frac{m}{4})^2}}$$
$$\geq \frac{m^2}{8} \frac{8z_{2i+1} - m}{\sqrt{1 + 4{z_{2i+1}}^2}}$$

By summing,

$$L(g_m) - L(g_m^h) \geq m^2 \sum_{i=0}^{16p+1} \left(\frac{z_{2i+1}}{\sqrt{1 + 4{z_{2i+1}}^2}} - \frac{z_{2i}}{\sqrt{1 + 4{z_{2i}}^2}} \right) - \frac{m^3}{8} \sum_{i=0}^{32p+3} \frac{1}{\sqrt{1 + 4{z_i}^2}}.$$

Since the function $f_1(x) = \frac{x}{\sqrt{1+4x^2}}$ is monotonically increasing and concave, one has

$$\sum_{i=0}^{16p+1} (f_1(z_{2i+1}) - f_1(z_{2i})) \geq \frac{1}{2} \sum_{i=0}^{32p+3} (f_1(z_{i+1}) - f_1(z_i))$$
$$\geq \frac{1}{2} (f_1(z_{32p+4}) - f_1(z_0))$$

Moreover, the function $f_2(x) = \frac{1}{\sqrt{1+4x^2}}$ is monotonically decreasing and convex. Thus the Riemann sum $\sum_{i=0}^{32p+3} \frac{m}{\sqrt{1+4{z_i}^2}}$ is bounded by the integral $\int_{\frac{1}{16}}^{\frac{19}{48}} f_2(x)\,\mathrm{d}x$. It follows that

$$L(g_m) - L(g_m^h) \geq \frac{m^2}{2} \left(f_1(\frac{19}{48} + \frac{m}{2}) - f_1(\frac{1}{16} + \frac{m}{2}) \right.$$
$$\left. - \frac{1}{8} \operatorname{arg\,sinh}(\frac{19}{24}) + \frac{1}{8} \operatorname{arg\,sinh}(\frac{1}{8}) \right).$$

Since $m \leq \frac{1}{12}$ for any $p \in \mathbb{N}$, we obtain

$$L(g_m) - L(g_m^h) > 0.066 m^2.$$

Eventually, for any $h = \frac{1}{(12(8p+3))^2}$, we have shown that

$$L(g) \geq L(g_m) \geq L(g_m^h) + 0.06h.$$

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